**Introduction to Vectors**

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## Vectors

Using graphical representation for any vectors makes the process of understanding them easier. The graphing for the vectors can be done [here](https://academo.org/demos/3d-vector-plotter/).

Vectors are written as or or . Every vector starts from the origin.

For two vectors and , the resultant vector . Graphically, we can think of this as following to its tip, and then following the direction of from that point.

Multiplication is used to stretch or squeeze a vector. For example, would follow the same direction as , but for twice the distance. Similarly, would follow 1.5 times the distance as , but in the opposite direction.

In reality, any vector being represented as actually means is the result when the vector is multiplied by the constant and then added with the vector multiplied by the constant . and point in the directions of the and coordinates by default, each with length 1. They are known as the basis vectors of the coordinate system. Thus, a vector can be generally represented as .

This sort of combination is known as a Linear Combination of Vectors. This is because, for any vectors and , if the associated constant of one is fixed and the associated constant of the other changes, the resulting vector moves along a straight line, thus making the combination linear.

## Span

The possible area that a resultant vector can cover is known as its span. For the mentioned case, if one vector is kept constant and the other varies, the span of the resultant vector is a straight line through the origin. If both vectors vary, the span of the resultant vector is a plane through the origin since all points on that plane are covered. Similarly, if there are three vectors involved and all three vary, the span is a 3D space. If all the vectors are zero however, the span is just the origin.

## Linear Transformation

A transformation is like a function in that it takes one vector and gives out another. Graphically, it seems to move the vector. Graphical representation can make understanding transformations much easier.

Take the vector for example. We know that by default points towards the positive -axis and points towards the positive -axis. This means the and . However, what happens if we change the way the basis vectors are defined? The vector exists in relation to and , so if we change the values of and , moves. Since was initially , to find the new direction of we must simply replace the initial and with the new ones.

Say our transformation shifts to and shifts to . Thus, . The transformation is often represented as where the first column represents the new and the second column represents the new .

Understanding this makes it much easier to understand why traditional transformations are represented as they are. For example, a counter clockwise rotation is represented by the matrix . This is because to achieve this transformation, must point to where is pointing, i.e. and must point to the opposite direction of where is pointing, thus . Similarly, would represent a rotation.

### Shearing

Shearing is a particular type of transformation. Here, either or remains the same while the other becomes slanted. If the resultant vector is imagined to be the edge of a stack of cards, would change as if the stack were pushed a little from the top edge to make it slanted. This sort of transformation is represented by for shearing with and for shearing with , where is called the shearing factor.

### Solving Equations

Simultaneous equations follow the logic of vectors. For example, the two equations and can be written as , where is the original vector. Thus, is the new and is the new . So . The question in reality is thus asking what the original vector was given a new vector and the transformation matrix.

## Dot Products

The dot product of two vectors and , is written as

Graphically, we can think of a dot product as the projection of one vector on another, and then multiplying the length of that projection with the length of the other vector. If the projection is in the opposite direction, then the dot product is negative. If the vectors are perpendicular to each other, then there can be no projection, thus the dot product is 0.

We know that for a vector , . Thus, , where is the angle between the two vectors.

There are two laws related to dot products, the Schwarz Inequality and the Triangle Inequality.

The Schwarz Inequality states that . This is obvious since we know that the maximum value of is , meaning the dot product of two vectors will always be less than or equal to the product of their lengths.

The Triangle Inequality states that . This is also obvious since the maximum distance a resultant vector can travel is the combined distance travelled by the two vectors it consists of.

Row Pictures and Column Pictures with 2D Vectors

Take the two following equations:

This can be written in matrix format as:

This is known as the Row Picture of the system. Taking the transformation matrix to be , the original vector to be and the resultant vector to be , we have .

Graphically, we can draw two straight lines representing the two equations. The point of intersection of the two lines is the coordinates .

The same system can be written as:

This is known as the Column Picture of the system. Graphically, we can use vectors to represent a column picture. Here, is the transformed and is the transformed . Since the resultant vector is , we can find the linear combination required to achieve that vector, which gives us and .

## Row and Column Pictures for 3D Vectors

The underlying process for Row and Column Pictures is the same in 3D vectors as it is in 2D vectors.

Take the following equations:

The Row Picture for this system is:

For Row Pictures, one key difference is that it is possible to represent a plane instead of just a line. Take the first equation. The value of is not defined, so the equation holds true for any value of . Thus, the solution for that particular equation is a 2D plane in 3D space. The solution for the entire system however, is still a single point where the planes/lines meet. For this example, there are two planes formed by the first and third equations, which will intersect to form a line. The intersection point of that line and the line formed by the second equation is the solution.

The Column Picture for this system is:

This is solved in exactly the same way as in 2D vectors.

Can be solved for any ?

This is possible if the columns are random and there is contribution to all axes. For a matrix like , where the third column is just a multiple of the first, there is no new contribution by the third column. Completely independent and random columns are needed. This particular matrix would form a plane instead of a point. If two columns were the multiples of one, then a straight line would be formed.